SOME CONSIDERATIONS ON THE RELIABILITY IN THE PROBLEMS OF OPTIMAL CONTROL

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In this paper we formulate the problem of optimal control with the reliability of the controlling system taken into account (by reliability we mean the probability of the non-interrupted performance.

In the first part (Sect. 1-3) we consider the case when the reliability is specified. Taking into account the influence of the mode of performance of the system on the probability of failure, we formulate the variational problem for the case of a non-stationary Poisson type sequence of failures. The formulated problem is then investigated using L.S. Pontriagin's method and applied to the case of the delivery of maximum payload during the motion of a body of variable mass when the thrust of the jet (Sect. 2) and the exhaust velocity are limited (Sect. 3).

Analytical solutions of particular problems for the case of limited thrust are obtained in Sect. 2.

In the second part (Sections 4 and 5) we consider the problem of determining the optimal probability of failure-free performance of a rocket engine. Here, the following averaged characteritics (mathematical expectations) as maximum payload (Sect. 4) or the minimal cost of performing a manoeuvre (Sect. 5) are used as criteria of optimality.

The latter formulation refers to the case when such a manoeuvre has to be repeated a number of times. Both the above criteria are compared in Sect. 5.

1. Formulation of the variational problem. When the optimal control program is realised, the controlled system can be affected by random factors causing failures of the system. The probability of failures can depend on time, phase coordinates, control parameters and control functions. In such cases, some definite reliability may have to be stipulated in order to construct the optimal control program. This may, in turn, alter the character of the optimal control substantially.

In this former paper [1], the author considers one of the problems of optimal control in which random destructive processes act on the system. There however, the necessity of assuming some specified reliability did not manifest itself – instead some characteristics* which were averaged according to a definite method, were minimised. Moreover, the probability of failure was assumed to be independent of control functions.

Paper [3] investigates the problem when the duration of the controlling action is specified. If the dependence of the probability of failure-free performance on its duration is assumed known, then the specified duration of performance can be treated as the specified reliability. In this case however, the influence of phase coordinates and control functions which are both time-dependent on the probability of failure, cannot be estimated.

In the present paper the problem is formulated as follows: Consider a dynamic system whose behavior is described by a system of equations of the type

$$x_i^{\cdot} = f_i(t, x_j, u_k, w_l) \qquad (i, j = 0, 1, \ldots, n;\ k = 1, \ldots, r;\ l = 1, \ldots, q) \quad (1.1)$$

where x_i are phase coordinates of the system, u_k are control functions, w_l are the constant control parameters and a dot $^{\cdot}$ denotes differentiation with respect to time t.

Boundary conditions are given with respect to x_i at the times $t = 0$ and $t = T$, and our aim is to secure the maximum (minimum) value of the control functional of the problem $x_0(T)$. We shall call this the initial variational problem.

As far as the failures are concerned, the dynamic system (1.1) is considered to be a single unit – i.e. failure of one element causes the failure of the entire system.

We also assume that when the failure occurs at any instant of time, then the problem is not fulfilled, and, that after the failure the system remains inoperative.

Our aim is to construct such a control program and to choose such parameter values, which would not only satisfy all the conditions given above, but would also secure a specified probability of failure-free performance of the system (i.e. reliability specified in advance).

We assume that the failures form an ordinary sequence without after-effects [4], and the word 'ordinary' means here that the probability of simultaneous occurrence of two or more failures, is nil. The absence of after-effects means, that the probability of failure over a certain interval of time depends only on the length of that interval and is not influenced by the previous intervals.

It is also assumed that the mean number of failures λ per unit time (density of the sequence)

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* In [1] a derivation of formulas for the average moments of failures was not rigorous from the point of view of the theory of probability. The author is grateful to R.V. Studnev for pointing out this shortcoming which was removed from [2] without altering the formulas.

$$\lambda = \lambda(t, x_j, u_k, w_l) \geqslant 0 \tag{1.2}$$

is known and dependent on time, phase coordinates, controlling functions and control parameters. Then, the probability R of the absence of failures over the interval [0,T] is equal [4] to

$$R = \exp\left(-\int_0^T \lambda\, dt\right) \tag{1.3}$$

where λ is calculated along the trajectory $\lambda = \lambda(t, x_j(t), u_k(t), w_l)$.

The relationship (1.3) expresses the condition of specified reliability $0 < R < 1$. This can be brought in line with the equations of the initial variational problem either by the classical methods where (1.3) becomes an isoperimetric condition

$$\int_0^T \lambda(t, x_j, u_k, w_l)\, dt = -\ln R \tag{1.4}$$

or by utilising the maximum principle, when (1.3) becomes a differential equation with the following boundary conditions

$$\Lambda^{\cdot} = \lambda(t, x_j, u_k, w_l), \quad \Lambda(0) = 0, \quad \Lambda(T) = -\ln R \tag{1.5}$$

The variable Λ will appear here as an additional phase coordinate of the system (1.1). We shall call the value $\Lambda_1 = \Lambda(T)$ a conditionally admissible mean value of the failures over the whole time. If $R = 1$ is assumed, then $\Lambda_1 = 0$ cannot be fulfilled, but when $R < 1$ and decreases, then the admissible number of failures increases (e.g. the interval $0.5 \leqslant R \leqslant 1$ corresponds to the interval $0.7 \leqslant \Lambda_1 \leqslant 0$).

Let us now denote the solution of the initial variational problem by $x_j^*(t)$, $u_k^*(t)$, and w_l^*. If we also find that

$$\Lambda_1^* = \int_0^T \lambda(t, x_j^*, u_k^*, w_l^*)\, dt \leqslant -\ln R \tag{1.6}$$

then the reliability R will obviously be secured and the condition (1.3) will be superfluous. The same situation arises when $T\lambda_{\max} \leqslant -\ln R$.

Now, assuming that the condition (1.6) is not fulfilled, let us write the equations of the problem of optimal control with the reliability specified by ((1.1) + (1.5))

$$\begin{gathered} x_i^{\cdot} = f_i(t, x_j, u_k, w_l), \quad x_i(0) = x_{i0}, \quad x_i(T) = x_{i1}, \\ \Lambda^{\cdot} = \lambda(t, x_j, u_k, w_l), \quad \Lambda(0) = 0, \quad \Lambda(T) = -\ln R, \end{gathered} \quad x_0(T) = \max(\min) \tag{1.7}$$

We see that in the above form, (1.7) represents the Mayer's variational problem. We shall now make two remarks concerning the generalisation of (1.7).

1°. Let the density of the sequence of failures λ depend also on the duration of performance of the system t_μ ($t_\mu \leqslant t$, compare (1.2)). Then, by [3], a differential equation should be incorporated into (1.7). Its boundary conditions would be*

$$t_\mu^{\cdot} = \delta, \qquad t_\mu(0) = 0, \qquad t_\mu(T) = \text{opt}$$

where $\delta(t) = 1$ or 0 is a relay control function equal to unity when the system is on, and to zero when the system is off.

This function should, by means of a multiplier, be incorporated into the controls u in (1.7), which assume zero value when the system is off.

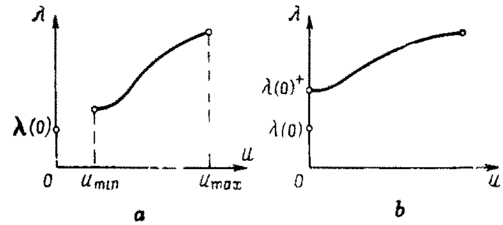

FIG. 1

2°. Let the control u be allowed to assume the values from the range $[u_{min}, u_{max}]$ and become zero in the off position. Density of the sequence of failures is equal here to $\lambda(u)$ and $\lambda(0)$, respectively (see Fig. 1*a*).

In this case, to represent the right-hand side of the last equation of (1.7) as an invariant type function, we shall use once more the relay control $\delta(t)$:

$$\dot{\Lambda} = \lambda(0) + [\lambda(u) - \lambda(0)]\,\delta$$

substituting at the same time $u\delta$ for u in the remaining equations.

The above approach can also be used when the density $\lambda(0)$ of the sequence of failures when the system is in an off position is different from the limit density $\lambda(0)^+$ when $u \to 0$ (see Fig. 1*b*)

Later we shall use this method to solve the problem of delivery of maximum payload when a body of variable mass moves in the gravity field (see [2]).

Two cases will be considered: that of restricted thrust and that of the limited exhaust velocity.

2. Reaction jet with restricted power. For the optimal (without considering reliability) modes of performance of a propulsion system with a limited power, we find that the maximum utilisation of power over the active intervals emerges as a characteristic feature. Moreover, if the propulsion system is ideally regulated (thrust and exhaust velocity not restricted) then passive intervals are absent from the optimal trajectory (see review [2]).

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* Here, contrary to [3], the final value of $t_\mu(T)$ is not given and should be selected so as to conform to the optimal trajectory.

We shall now define the probability (1.3) of successful completion of a manoeuvre. The density of the sequence of failures (1.2) we shall assume to be dependent on the mode of performance of the engine

$$\lambda = \lambda\ (t, N) \qquad (\partial\lambda/\partial N \geqslant 0,\ 0 \leqslant N(t) \leqslant 1) \tag{2.1}$$

where N is the power expressed in terms of the maximum power.

If we limit ourselves to the case of an ideally regulated propulsion system, then the problem of delivery of the maximum payload will separate, as before, into a static and a dynamic part; the character of the solution of the static part of the problem will, for example, be not altered in case of a single-stage device

$$G_\pi = (1 - \sqrt{\Phi})^2, \qquad G_\nu = \sqrt{\Phi} - \Phi, \qquad G_\mu = \sqrt{\Phi} \tag{2.2}$$

where G_π, G_ν and G_μ are the weights of the payload, engine and propellant, respectively, referred to the initial weight of the device. Φ is given by

$$\Phi = \frac{\alpha}{2g}\int_0^T \frac{a^2}{N}\,dt \qquad \left(J = \int_0^T \frac{a^2}{N}\,dt\right) \tag{2.3}$$

where α is the specific mass of the engine, g is the acceleration due to gravity, a is the acceleration due to the mass expelled, i.e. thrust divided by the mass flux out of the vehicle. The dynamic part of the problem reduces to the minimisation of the integral J in (2.3). An argument showing that $N\ (t) \equiv 1$ is optimal since N does not enter the equation of motion $\mathbf{r}^{\cdot\cdot} = a\mathbf{i} + \mathbf{g}$ ($\mathbf{r}$ is the radius vector, $\mathbf{g} = \mathbf{g}\ (\mathbf{r},\ t)$ is the gravity vector, $\mathbf{i}$ is the unit thrust vector), could be presented if the reliability was not taken into account. We cannot do this here, since (1.5) is added to the equation of motion, (1.5) determining the allowed number of failures, and λ is defined by (2.1). If N increases, then the expelled mass decreases (see the integral in (2.3)), if the acceleration to be achieved is kept constant ($a\ (t) =$ const). Density of the sequence of failures (2.1) also increases with N, hence we can expect the optimal program $N\ (t)$ to be different from $N\ (t) \equiv 1$.

Equations (1.7) of the variational problem of constructing the optimal trajectory when the reliability of the completion of a manoeuvre is given, consist of the equation describing the change of the integral (2.3), equations of motion and of the equations defining the allowed number of failures with the corresponding boundary conditions

$$\begin{array}{lll}
J^{\cdot} = a^2 / N, & J\ (0) = 0, & J\ (T) = \min \\
\mathbf{r}^{\cdot} = \mathbf{v}, & \mathbf{r}\ (0) = \mathbf{r}_0, & \mathbf{r}\ (T) = \mathbf{r}_1 \\
\mathbf{v}^{\cdot} = a\mathbf{i} + \mathbf{g}, & \mathbf{v}\ (0) = \mathbf{v}_0, & \mathbf{v}\ (T) = \mathbf{v}_1 \\
\Lambda^{\cdot} = \lambda\ (t, N), & \Lambda\ (0) = 0, & \Lambda\ (T) = -\ln R
\end{array} \tag{2.4}$$

$$(0 \leqslant a\ (t) < \infty,\ 0 \leqslant N\ (t) \leqslant 1,\ |\,\mathbf{i}\ (t)\,| \equiv 1)$$

Optimal laws of change of the acceleration due to reaction $a\ (t)$, of the power $N\ (t)$ and of the direction of the thrus vector $\mathbf{i}\ (t)$ should, by the maximum principle, secure at any time the absolute maximum of the function H (minimum of $J(T)$ is sought)

$$H = -a^2/N + a(\mathbf{p}_v \cdot \mathbf{i}) + \lambda(t, N)p_\lambda + (\mathbf{p}_r \cdot \mathbf{v}) + (\mathbf{p}_v \cdot \mathbf{g}) \quad (2.5)$$

where the impulses $\mathbf{p}_r$, $\mathbf{p}_v$, and $\mathbf{p}_\lambda$ satisfy the equations

$$\mathbf{p}_r^{\cdot} = -\frac{\partial H}{\partial \mathbf{r}} = -\frac{\partial}{\partial \mathbf{r}}(\mathbf{p}_v \cdot \mathbf{g}), \qquad \mathbf{p}_v^{\cdot} = -\frac{\partial H}{\partial \mathbf{v}} = -\mathbf{p}_r, \quad p_\lambda^{\cdot} = -\frac{\partial H}{\partial \Lambda} = 0 \quad (2.6)$$

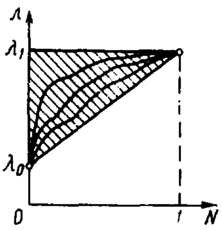

FIG. 2

The impulse corresponding to the phase coordinate J is assumed to be identically equal to minus unity, since we seek the minimum of $J(T)$ and J does not appear in the right hand sides of the above equations $(\partial H / \partial J = 0)$.

Maximum of H with respect to $\mathbf{i}$ and a is reached when

$$\mathbf{i} = \mathbf{p}_v / p_v, \qquad a = {}^1/_2 p_v N \qquad (p_v = |\mathbf{p}_v|) \quad (2.7)$$

and after that, the part of H which is dependent on N, assumes the form

$$H_N = {}^1/_4 p_v^2 N + \lambda(t, N)p_\lambda \quad (2.8)$$

which can be used as a basis for determining the sign of the impulse p_λ = const. (see (2.6)).

Let us assume that if $p_\lambda \geqslant 0$, then the maximum of H occurs when $N = 1$ (since by (2.1), $\partial\lambda / \partial N \geqslant 0$). This means that the optimal trajectory will not change on neglecting the reliability.

Now, when we formulated the general problem (1.7), we assumed that the condition (1.6) was not fulfilled and that a given reliability R could not be secured on such a trajectory. Hence

$$p_\lambda < 0 \quad (2.9)$$

We shall note two properties of optimal trajectories with reliability, which result from the condition of maximum of (2.8) in terms of N.

1°. For all functions $\lambda(t, N)$ from (2.1) which satisfy the conditions

$$\lambda(t, N) \geqslant \lambda_0 + (\lambda_1 - \lambda_0)N, \qquad \lambda(t, 0) = \lambda_0(t), \qquad \lambda(t, 1) = \lambda_1(t) \quad (2.10)$$

(see the shaded region on Fig. 2), the control $N(t)$ is the limiting control and the optimal trajectories coincide with the values of the functional $J(T)$.

Indeed, if we substitute a linear function $\lambda(N)$ from the inequality (2.10) into (2.8), then the resulting function H_N

$$H_{N\,\mathrm{maj}} = \lambda_0 p_\lambda + [{}^1/_4 p_v^2 + (\lambda_1 - \lambda_0)p_\lambda]N$$

will, by (2.9), be an upper bound for all H_N containing $\lambda(t, N)$, satisfying (2.10)

$$H_N(N) \leqslant H_{N\,\mathrm{maj}}(N), \qquad H_N(0) = H_{N\,\mathrm{maj}}(0), \qquad H_N(1) = H_{N\,\mathrm{maj}}(1)$$

Hence, $\max H_N \leqslant \max H_{N\,\mathrm{maj}}$, $H_{N\,\mathrm{maj}}$ however is a linear function of N, and its maximum is reached at the boundary points

$$N = 1 \quad \text{as} \quad p_v^2 > -4(\lambda_1 - \lambda_0)p_\lambda, \qquad N = 0 \quad \text{as} \quad p_v^2 < -4(\lambda_1 - \lambda_0)p_\lambda \quad (2.11)$$

At these points functions H_N and $H_{N\,\mathrm{maj}}$ coincide, therefore the maximum H_N for all λ from (2.10) will occur at the above points. Hence, the optimal control $N(t)$ is the limit of (2.11) and the intermediate values of λ will be absent from the problem.

At the point $N = 1$ and $N = 0$, all $\lambda(t, N)$ from (2.10) assume the same values $\lambda_1(t)$ and $\lambda_0(t)$. This means that the optimal trajectories and the magnitudes of the functionals $J(T)$ will be identical.

2°. If $\partial\lambda / \partial N = 0$ at $N = 0$, then the optimal trajectory does not contain any passive intervals. In this case the partial derivative of (2.8) with respect to N is

$$\partial H_N / \partial N = {}^1\!/_4 p_v^2 + p_\lambda \partial\lambda / \partial N$$

and will, at the point $N = 0$, be always positive (except for the isolated moments when $p_v = 0$). Hence, the optimal value of N which guarantees the maximum of (2.8) is always positive, i.e. passive intervals are absent.

Next we shall consider a particular case of (2.1)

$$\lambda = \lambda(N) = \lambda_{\max} N^n \qquad (\lambda_{\max} = \text{const}, \ n > 0) \qquad (2.12)$$

(see Fig. 3). For all $0 < n \leqslant 1$ (shaded region on Fig. 3), the optimal control $N(t)$ is defined by (2.11), where $\lambda_0 = 0$, $\lambda_1 = \lambda_{\max}$, and the property 1° holds. For $n > 1$, we have the property 2°, and the optimal control $N(t)$ is

$$N = 1 \quad \text{as} \quad p_v^2 \geqslant -4np_\lambda\lambda_{\max}, \quad N = \left(-\frac{p_v^2}{4np_\lambda\lambda_{\max}}\right)^{1/(n-1)} \quad \text{as} \quad p_v^2 \leqslant -4np_\lambda\lambda_{\max} \quad (2.13)$$

i.e. intervals appear, over which the performance is variable and less than maximum.

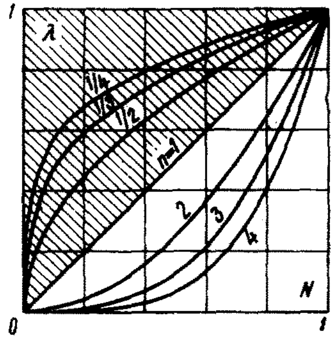

FIG. 3

The results obtained above do not depend on the type of the dynamic manoeuvre (i.e. on $\mathbf{r}_0$, $\mathbf{v}_0$, $\mathbf{r}_1$, $\mathbf{v}_1$ and $\mathbf{g}(\mathbf{r}, t)$). In order to complete the solution of the variational problem (2.4) we shall presuppose the simplest manoeuvres — one dimensional motions in the force-free field ($\mathbf{g} = 0$): translation between two points at rest and attaining the given velocity.

In the absence of gravity, equations (2.6) can be integrated directly

$$p_r = \text{const}, \qquad p_v = p_{v0} - p_r t \tag{2.14}$$

For the problem of translation between two points at rest, we have the following boundary conditions

$$r(0) = 0, \quad v(0) = 0, \quad r(T) = l, \quad v(T) = 0$$

The problem is symmetrical with respect to $t = \frac{1}{2}T$ (with the accuracy up to the instant of reversal of the direction of thrust), consequently we can restrict ourselves to considering only the first half of the motion $(0 \leqslant t \leqslant {}^1/_2T)$, writing the boundary conditions as follows

$$r(0) = 0, \quad v(0) = 0, \quad r({}^1/_2T) = {}^1/_2 l, \quad v^{\cdot}({}^1/_2T) = 0$$

and doubling the value obtained for the functional $J(T) = 2J(\frac{1}{2}T)$. Since $v^{\cdot}({}^1/_2T) = 0$, consequently $p_v({}^1/_2T) = 0$, i.e. $p_v = p_{v0}(1 - 2t/T)$.

Substitution of (2.7), (2.11) or (2.13) into (2.4) to obtain the optimal controls with the function $p_v(t)$, makes it possible to integrate (2.4) by parts. The unknown constsnts p_{v0} and p_λ are found from the final conditions

$$r({}^1/_2T) = {}^1/_2 l, \quad \Lambda({}^1/_2T) = -{}^1/_2 \ln R$$

If the reliability is not fixed $(\Lambda(T) = \text{opt})$, then $p_\lambda = 0$ and we have the following solution (see [2]):

$$N^*(t) = 1, \qquad a^*(t) = 6(l/T^2)(1 - 2t/T), \qquad J^*(T) = 12(l^2/T^3) \tag{2.15}$$

With R given, the solution will depend on the parameter

$$\varkappa = -\ln R / T\lambda_{max} \qquad (0 < \varkappa \leqslant 1) \tag{2.16}$$

which represents the ratio of allowed number of failures $\Lambda(T) = -\ln R$ to the maximum possible number of them, which is $\Lambda_{max} = T\lambda_{max}$.

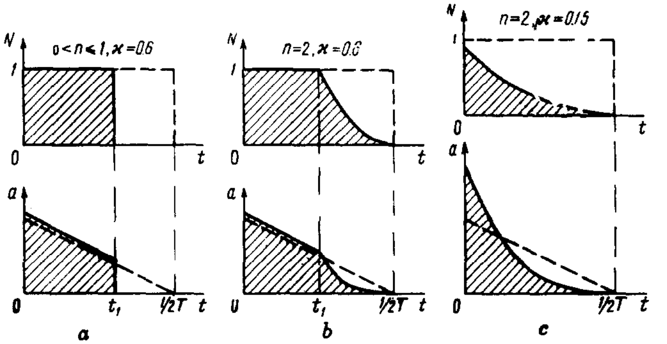

FIG. 4

Finally we obtain the solution in the following form

For $0 < n \leqslant 1,\ 0 < \varkappa \leqslant 1$ (**Fig. 4a**, comp. [3])

For $0 \leqslant t \leqslant t_1$

$$N(t) = 1, \quad a(t) = a_0(1 - 2t/T) \qquad a_0 = \frac{6l}{T^2}\left[1 - \left(1 - 2\frac{t_1}{T}\right)^3\right]^{-1}, \quad t_1 = \frac{1}{2}\varkappa T$$

when $t_1 \leqslant t \leqslant {}^1/_2 T$

$$N(t) = 0, \quad a(t) = 0$$

$$J(T) = 12\,(l^2 / T^3)\,[1 - (1 - \varkappa)^3]^{-1}$$

For $n > 1,\ (n-1)/(3n-1) \leqslant \varkappa \leqslant 1$ (**Fig. 4b**) (2.17)

when $0 \leqslant t \leqslant t_1$

$$N(t) = 1, \quad a(t) = a_0(1 - 2t/T), \qquad a_0 = \frac{6l}{T^2}\left[1 - \frac{2}{3n-1}\left(1 - 2\frac{t_1}{T}\right)^3\right]^{-1}$$

when $t_1 \leqslant t \leqslant {}^1/_2 T$

$$N(t) = \left(\frac{1 - 2t/T}{1 - 2t_1/T}\right)^{2/(n-1)}, \quad a(t) = a_0\frac{(1 - 2t/)T^{(n+1)/(n-1)}}{(1 - 2t_1/T)^{2/(n-1)}}, \quad t_1 = \frac{3n-1}{4n}\left(\varkappa - \frac{n-1}{3n-1}\right)T$$

$$J(T) = 12\left(\frac{l^2}{T^3}\right)\left[1 - \frac{1}{4}\,\frac{(3n-1)^2}{n}(1 - \varkappa)^3\right]^{-1}$$

For $n > 1,\ 0 < \varkappa \leqslant (n-1)/(3n-1)$ (**Fig. 4c**)

$$N(t) = \left(\varkappa\frac{3n-1}{n-1}\right)^{1/n}\left(1 - 2\frac{t}{T}\right), \qquad a(t) = \frac{6l}{T^2}\,\frac{3n-1}{3(n-1)}\left(1 - 2\frac{t}{T}\right)^{(n+1)/(n-1)}$$

$$J(T) = 12\,\frac{l^2}{T^3}\,\frac{1}{3}\left(\frac{3n-1}{n-1}\right)^{(n-1)/n\,3}\varkappa^{-1/n}$$

Fig. 4a to c gives the comparison of changes of power N and acceleration due to thrust. Dotted lines represent the laws (2.15) which are optimal in absence of failures, while solid lines represent (2.17).

On Fig. 5, solid represent the relation J/J^* versus $\varkappa$ where J is taken from (2.17), J^* from (2.15) and $\varkappa$ from (2.16), for various values of n shown in (2.12). Increase in reliability (decrease in $\varkappa$) with other conditions kept unchanged, leads to the increase of the functional J. Corresponding decrease in the payload can be calculated from (2.2) and (2.3).

The difference between the curves $0 < n \leqslant 1$ and those for $n = 0, 2, 3, 4$, gives some idea of the advantage which can be gained by using the intermediate power values.

If at $n > 1$ the condition of the fixed reliability could be fulfilled only by switching the engine off, then the functional would have the same value for all $\lambda(N)$ (the curve $0 < n \leqslant 1$). Program (2.13) makes it possible to lower considerably the value of the functional.

For the cases when a certain velocity has to be reached (velocity increment), the boundary conditions are

$$r(0) = 0, \qquad v(0) = 0, \qquad r(T) = \text{opt}, \qquad v(T) = \Delta v$$

Here the value of $r(T)$ is not given, hence $p_r(T) = 0$ and, according to (2.14),

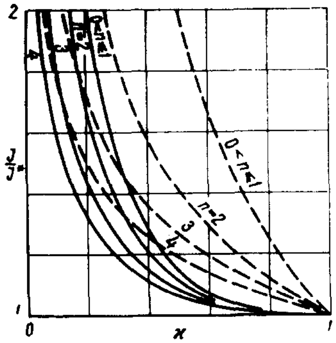

FIG. 5

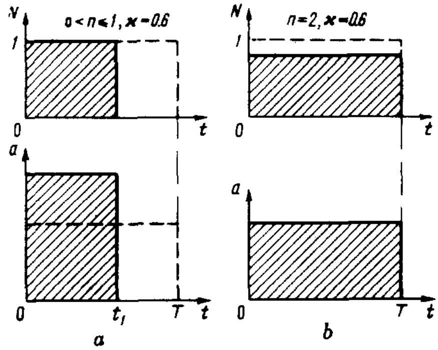

FIG. 6

$p_v(t) = p_{v0}$. Hence power N and acceleration a are constant over the active intervals (see (2.13), (2.11), (2.7)). If reliability is disregarded ($p_\lambda = 0$) , then passive intervals are absent from the tajectory and the following solution is obtained (see [2]):

$$N^*(t) = 1, \qquad a^*(t) = \Delta v / T, \qquad J^*(T) = \Delta v^2 / T \tag{2.18}$$

If the value of the index n in (2.12) falls within the range $0 < n \leqslant 1$, then the passive integrals appear. Their number and distribution do not influence the functional of the problem, and their total duration is chosen from the condition of a given reliability $\Lambda(T) = -\ln R$. When $n > 1$, then by 2° passive intervals are absent and the condition $\Lambda(T) = -\ln R$ can be fulfilled only by reducing N.

Performing all the calculations we obtain
when $0 < n \leqslant 1,\ 0 < \varkappa \leqslant 1$ (Fig. 6*a*, comp. [3])

$$\begin{array}{lll} N(t) = 1, & a(t) = \Delta v / t_1 & \text{for } 0 \leqslant t \leqslant t_1 \\ N(t) = 0, & a(t) = 0 & \text{for } t_1 \leqslant t \leqslant T \end{array} \qquad (t_1 = \varkappa T)$$

$$J(T) = (\Delta v^2 / T)\, \varkappa^{-1} \tag{2.19}$$

when $n > 1,\ 0 < \varkappa \leqslant 1$ (Fig. 6*b*)

$$N(t) = \varkappa^{1/n}, \qquad a(t) = \Delta v / T, \qquad J(T) = (\Delta v^2 / T)\, \varkappa^{-1/n}$$

Dependence of the functionals (2.19) and (2.18) on $\varkappa$ and n is shown on Fig. 5 by means of broken lines. We can see that for the case of translation between two points at rest (solid lines), the requirement of fixed reliability leads to much smaller increase in the value of the functional, than for the case of velocity increment.

In order to estimate the corresponding loss in the payload, we must use the first relation from (2.2). Fig. 7 shows an example ($\Phi^* = (\alpha / 2g)\ J^* = 0.1,\ 0 < n \leqslant 1$) of the dependence of the payload on the reliability for various values of the maximum possible number of failures $\Lambda_{max} = T\lambda_{max} = 0.2, 0.5, 1, 2$ (here solid lines refer to translation between two points at rest, broken lines to the velocity increment).

Horizontal parts of curves on Fig. 7 correspond to the values of R, for which the inequality (1.6) also holds.

3. Limited exhaust velocity. In the problem on the maximum payload for propulsion system with the limited exhaust velocity (thermal rocket motors) we find it convenient to use, over the active intervals (without considering reliability), the notion of maximum exhaust velocity.

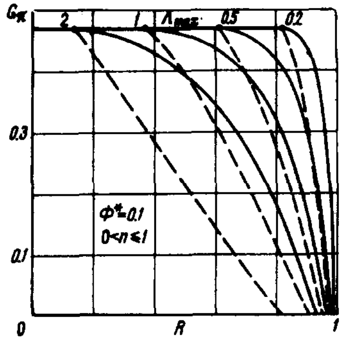

FIG. 7

Let us assume that the rise in the exhaust velocity (i.e. rise in temperature) leads to the increase in λ (1.2) (also compare (2.1))

$$\lambda = \lambda\,(t,\, V) \qquad (\partial\lambda\,/\,\partial V \geqslant 0,\ 0 \leqslant V \leqslant 1) \quad (3.1)$$

(in this and subsequent arguments exhaust velocity V is expressed in terms of maximum exhaust velocity V_0). If the reliability is fixed (as in case of restricted power) then some intermediate values of exhaust velocity may become optimal.

We shall consider the maximum thrust P_0 and the maximum exhaust velocity as known. Then the problem on the maximum payload becomes a problem on the maximum final mass, and equations of the variational problem (1.7) can now be written (comp. (2.4), thus

$$\begin{aligned}
&G^{\cdot} = -\mu\,(P\,/\,V)\,\delta, && G\,(0) = 1, && G\,(T) = \max \\
&\mathbf{r}^{\cdot} = \mathbf{v}, && \mathbf{r}\,(0) = \mathbf{r}_0, && \mathbf{r}\,(T) = \mathbf{r}_1 \\
&\mathbf{v}^{\cdot} = a_0\,(P\,/\,G)\,\mathbf{i}\delta + \mathbf{g}, && \mathbf{v}\,(0) = \mathbf{v}_0, && \mathbf{v}\,(T) = \mathbf{v}_1 \\
&\Lambda^{\cdot} = \lambda_0\,(t) + [\lambda\,(t,\,V) - \lambda_0\,(t)]\,\delta, && \Lambda\,(0) = 0, && \Lambda\,(T) = -\ln R
\end{aligned} \quad (3.2)$$

$$(0 \leqslant P\,(t) \leqslant 1,\quad 0 \leqslant V\,(t) \leqslant 1,\quad \delta\,(t) = 1 \ \text{ or } \ 0,\ |\,\mathbf{i}(t)\,| \equiv 1)$$

Here G is the instantaneous weight (referred to the initial weight G_0), $\mu = gP_0\,/\,G_0V_0$ is the corresponding mass flux out of the vehicle (a known parameter), $a_0 = gP_0\,/\,G_0$ is the initial acceleration due to thrust (a known parameter). Thrust P and exhaust velocity V are referred to their maximum values P_0 and V_0.

It is assumed that when the engine is off ($\delta = 0$), the density of the sequence of failures becomes $\lambda_0(t)$ (when $\lambda_0 = 0$, then $\lambda_0(t)$ ($\lambda_0\,(t) < \lambda\,(t,\,V)$. To reflect this fact in (3.2), we have used the case 2° from Sect. 1.

If it was known that the thrust P did not assume intermediate values $0 < P \leqslant 1$ on any optimal trajectory (so called special condition) then the function δ could be omitted and replaced in the last equation by P. In our case, however, this is not immediately obvious.

To analise the structure of the optimal control, we must investigate the absolute minimum of the function H (compare (2.5)) (we seek the maximum of $G\,(T)$) with reference to $\mathbf{i}$, P, δ, and V.

$$H = \left\{\left[- p_\mu \frac{\mu}{V} + \frac{a_0}{G}(\mathbf{p}_v \cdot \mathbf{i})\right] P + p_\lambda [\lambda (t, V) - \lambda_0 (t)]\right\} \delta + \\ + \lambda_0 (t) p_\lambda + (\mathbf{p}_r \cdot \mathbf{v}) + (\mathbf{p}_v \cdot \mathbf{g}) \tag{3.3}$$

Impulses $\mathbf{p}_r$, $\mathbf{p}_v$, and p_λ are defined, as before, by (2.6), while the impulse p_μ is given by

$$p_\mu = - \partial H / \partial G = (\mathbf{p}_v \cdot \mathbf{i}) a_0 G^{-2} P \delta, \qquad p_\mu (T) = -1 \tag{3.4}$$

From the minimum of (3.3) with respect to $\mathbf{i}$ it follows (comp. (2.7)), that

$$\mathbf{i} = - \mathbf{p}_v / p_v \tag{3.5}$$

Using an argument analogous to that in the Sect. 2 (see (2.9)) we can show that

$$p_\mu (t) \leqslant 0, \qquad p_\lambda > 0 \tag{3.6}$$

If we now assume that at some instant of time $p_\mu (t) > 0$, then minimum of H implies $V = 0$ (irrespective of the sign of p_λ) and $P = \delta = 1$. Now, $P = 1$ when $V = 0$ is physically inadmissible, hence $p_\mu (t) \leqslant 0$ everywhere. If we now put $p_\lambda < 0$, we shall obtain $V = 1$ everywhere and the moment at which the engine is turned off will occur later than in case of $p_\lambda = 0$. At the same time the condition of fixed reliability cannot be fulfilled (since it is assumed that the inequality (1.6) does not hold).

We shall now show that the thrust cannot assume intermediate values $0 < P < 1$ on the optimal trajectory. Since P occurs in (3.3) in the linear form, optimal intermediate values of P may appear when the coefficient of P becomes equal to zero within some interval of time (special conditions). In this case however $\delta = 0$ since $p_\lambda > 0$ (see (3.6)) i.e. the thrust should be equal to zero.

The above is valid for any manoeuvre in an arbitrary gravitational field. Having proved the absence of special conditions in the structure of the optimal control, we shall exclude the control function $\delta (t)$ from (3.2) to (3.4): we shall replace all $P\delta$ by P and substitute P for any δ appearing without P.

Then the part of H dependent on the controls V and P will, together with (3.4), become

$$H_{V,P} = [H_V - p_v a_0 / G - \lambda_0 (t) p_\lambda] P, \qquad H_V = - \mu p_\mu / V + \lambda (t, V) p_\lambda \tag{3.7}$$

From the condition of minimum of H_V with respect to V, follows

$$V = 1 \quad \text{if} \quad \lambda (t, V) \geqslant \lambda (t, 1) + (\mu p_\mu / p_\lambda)(1 / V - 1) \tag{3.8}$$

If the condition (3.8) is not fulfilled, then there exists an optimal intermediate value $0 < V < 1$, which can be found from

$$V^2 \partial \lambda / \partial V = - \mu p_\mu / p_\lambda \qquad (0 < V < 1) \tag{3.9}$$

Null values of V are absent from the optimal control. When

$$\lambda (t, V) \geqslant \lambda (t, 1) + (\mu, p_{\mu 0} / p_\lambda) (1 / V - 1)$$

the condition (3.8) will be valid over the whole trajectory, since $p_\mu \leqslant 0,\ p_\mu \leqslant 0$ (see (3.4) to (3.6)). In this case the exhaust velocity will never assume any intermediate values.

Optimal control in terms of thrust can, as we showed before, be only a limiting control. The sequence of active and passive intervals on the trajectory is determined by

$$P = 1 \quad \text{if} \quad H_V - p_v a_0 G^{-1} - \lambda_0 p_\lambda < 0, \quad P = 0 \quad \text{if} \quad H_V - p_v a_0 G^{-1} - \lambda_0 p_\lambda > 0 \quad (3.10)$$

which is obtained from the condition of absolute minimum of the function (3.7) with respect to P.

Let us now consider the case of limited acceleration a (in place of limited thrust). Here, the problem on the maximum final mass G_1 reduces to finding the minimum of the functional

$$J = \int_0^T \frac{a}{V} dt \qquad (0 \leqslant V \leqslant 1,\ G_1 = G_0 e^{-J / V_0}) \tag{3.11}$$

(so called characteristic velocity, compare with (2.2) and (2.3)).

The absence of intermediate values $0 \leqslant a \leqslant a_0$ in the system of optimal control can again be proved in the manner similar to that of the previous case. To do this, we can use the (3.2) form of the equations of the variational problem (1.7), replacing its first and last equations

$$J^{\cdot} = (a_0 / V)\, \delta, \quad (J(0) = 0,\ J(T) = \min), \qquad \mathbf{v}^{\cdot} = a_0 \delta \mathbf{i} + \mathbf{g}$$

The optimal control of thrust will be determined from the first relation of (2.7). Controls $V(t)$ and $\delta(t)$ can be found from the condition of the maximum of the function (compare (3.7))

$$H_{V,\delta} = [H_V + p_v a_0 - \lambda_0 (t)\, p_\lambda]\, \delta \qquad (H_V = -a_0 / V + \lambda (t, V)\, p_\lambda, \quad p_\lambda^{\cdot} < 0)$$

To find the optimal exhaust velocity we shall use (3.8) and (3.9) in which however the substitution $\mu p_\mu = -a_0$ should be made.

Moreover, if $\partial \lambda / \partial t = 0$, then the optimal exhaust velocity colinear with the direction of motion will be constant. Times at which the engine will be on ($\delta = 1$) or off ($\delta = 0$) can be found from the relations

$$\delta = 1 \ \text{when}\ H_V + p_v a_0 - \lambda_0 (t)\, p_\lambda > 0, \qquad \delta = 0 \ \text{when}\ H_V + p_v a_0 - \lambda_0 (t)\, p_\lambda < 0$$

analogous to (3.10).

4. Criterion of weight. Existence of the optimal probability of completing a manoeuvre can be shown by means of the following argument. The increase of reliability can, as shown in the sections 2 and 3, be achieved at the expense of cutting down the payload (this is equivalent to the increase in cost). Nevertheless, the above effect is accompanied by the increase of the percentage of successful attempts of completing a manoeuvre. At the same time the optimal probability of completing a manoeuvre will always lie within the interval [0, 1] since, when the reliability is nil, the percentage of successful attemprs is also nil, while, when the value of the reliability is very much smaller than unity, the payload becomes equal to zero.

Let us assume that n manoeuvres successfully complete required $(m \geqslant n)$ attempts which failed due to some adverse external factors. Then, the amount of payload successfully delivered to the terminal point of the trajectory per one attempt, will be

$$G_\pi^{(n,m)} = \frac{n}{m} G_\pi \qquad (4.1)$$

where G_π is the payload, the delivery of which was the aim of each attempt.

We shall classify an attempt as unsuccessful if failures occur during approach to the final orbit or during the execution of a necessary manoeuvre. First of the above cases is characterised by R_0 which is the reliability of placing the rocket successfully in the initial orbit, while the other is characterised by the reliability R of the performance of the rocket during flight.

We assume that R_0 and the mean number of failures λ of the engine per unit time, the latter defining the reliability R (see (1.2) (1.3)), are both known.

From the above probabilistic characteristics we can derive the mathematical expectation (4.1) of the payload delivered successfully per one attempt

$$\langle G_\pi \rangle = R_0 R G_\pi \qquad (4.2)$$

(since $\langle m \rangle = n / R_0 R$ according to the rule of multiplication of probabilities). The maximum (4.2) is taken as a criterion of optimality, and the corresponding variational problem is presented in form of the Mayer problem

$$\begin{array}{lll} \langle G_\pi \rangle^\cdot = -R_0 e^{-\Lambda}(\lambda G_\sigma + q), & \langle G_\pi \rangle_{t=0} = G_\sigma(0) = 1 - G_\varkappa, & \langle G_\pi \rangle_{t=T} = \max \\ G_\sigma^\cdot = -q, & & G_\sigma(T) = \text{opt} \\ \mathbf{r}^\cdot = \mathbf{v}, & \mathbf{r}(0) = \mathbf{r}_0, & \mathbf{r}(T) = \mathbf{r}_1 \\ \mathbf{v}^\cdot = P\mathbf{i} / (G_\sigma + G_\varkappa) + \mathbf{g}, & \mathbf{v}(0) = \mathbf{v}_0, & \mathbf{v}(T) = \mathbf{v}_1 \\ \Lambda^\cdot = \lambda(t, q, P, \ldots), & \Lambda(0) = 0, & \Lambda(T) = \text{opt} \end{array} \qquad (4.3)$$

Phase coordinates of the problem are $\langle G_\pi \rangle$, G_σ, r, v, Λ (G_σ is the total weight of the payload G_π and the propellant G_μ; $G_\varkappa$ is the weight of the engine, all weights are expressed in terms of the total initial weight). Control functions are P, q and $\mathbf{i}$ (q is the mass exhausted, P and q are referred to the initial mass).

To present the system (4.3) more fully, we must say something more definite about the formula for the weight of the engine, i.e. we must show the connection between its weight G_x and the limit values of control parameters, on which the thrust and the amount of mass exhausted depend.

In this case it is convenient to replace P and q assumed to be independent control functions, by parameters on which some constraints have been placed [2].

Variational problem (4.3) differs from (1.7) in the final condition referring to the phase coordinate Λ. In (1.7) a final value of Λ is given ($\Lambda(T) = -\ln R$), while in (4.3) it is chosen from the condition of optimality.

Remarks 1° and 2° made in Sect. 1 about (1.7) are valid for (4.3), and the properties of optimal controls established in Sections 2 and 3 still hold.

If the solution of the variational problem on maximum payload when reliability (i.e. the dependence of G_π on R) is given, is known, then the functional (4.2) becomes simply R, and the problem (4.3) is reduced to finding the optimal reliability R from the condition of maximum (4.2).

The procedure of constructing the optimal controls (4.3) will not be given here, since it is analogous to that in Sections 2 and 3.

We shall limit ourselves to solving an example on the ideally regulated propulsion system of limited power. In Sect. 2 we have obtained corresponding solutions of the problem on maximum payload with fixed reliability

$$R = e^{-\varkappa\Lambda_{max}} \tag{4.4}$$

where $\Lambda_{max} = T\lambda_{max}$ is the maximum possible number of failures over the whole time of flight T (λ_{max} is here the maximum density of the sequence of failures).

The parameter Λ_{max} is determined by the conditions of the problem while the parameter $\varkappa$ is chosen so as to satisfy the condition of reliability given by (4.4).

In Sect. 2 we have illustrated in more detail the dependence of the density of the sequence of failures on controlling functions (see (2.12)) by giving analytical relationships between $J(\varkappa)$ which was the minimal value of the problem (2.4) and the parameter $\varkappa$. In the present case, as we have already mentioned, the functional (4.2) of the problem under consideration, will be the function of the parameter $\varkappa$

$$\langle G_\pi \rangle = R_0 e^{-\varkappa\Lambda_{max}} \left[1 - \left(\frac{\alpha}{2g} J(\varkappa)\right)^{1/2}\right]^2 \tag{4.5}$$

where $J(\varkappa)$ is determined by (2.17) and (2.19) when $0 < \varkappa \leqslant 1$ and where $J(\varkappa) = J(1)$ when $\varkappa \geqslant 1$. ((2.17) corresponds here to the translation between two points at rest in the force-free field, while (2.19) corresponds to the case of velocity increment in the force-free field, see also Fig. 5).

Fig. 8 shows an example of the dependence of the mean value of successfully delivered payload (4.5) on (4.4). Calculations were made for $\Phi^* = (\alpha / 2g)\ J\ (1) = 0.25$, $R_0 = 1$, $0 < n \leqslant 1$ (see (2.12)) for various values of $\Lambda_{mas} = 0.2, 0.5, 1, 2$.

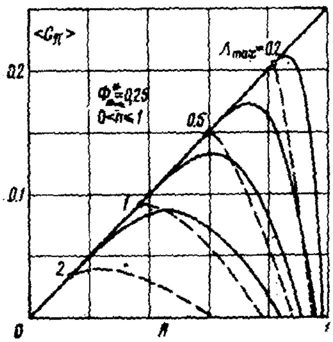

FIG. 8

Solid lines here refer to the problem of translation between two points at rest, broken ones to the problem of velocity increment. Points of contact of the curves with the straight line $\langle G_\pi \rangle = R\ (1 - \sqrt{\Phi^*})^2$, corresponding to $\varkappa = 1$ are also shown on the graph. The curves possess distinct maxima which are reached either at $0 < \varkappa < 1$ when $\partial \langle G_\pi \rangle / \partial \varkappa = 0$, or at $\varkappa = 1$.

For the problem on translation between two points at rest, maximum payload and optimal probability of fulfilling a manoeuvre are considerably larger than for the problem on the velocity increment. This is illustrated on Fig. 9 a and b (optimal reliability R as function of

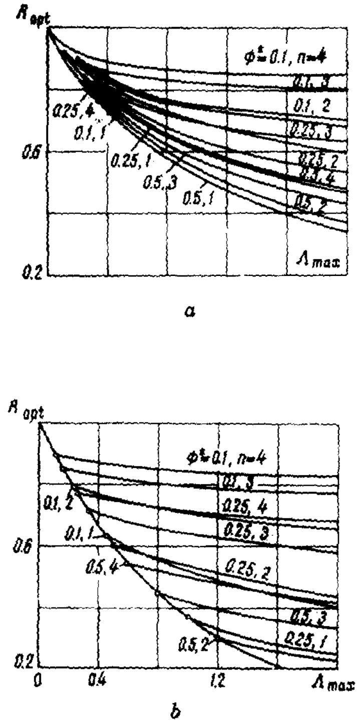

FIG. 9a, b

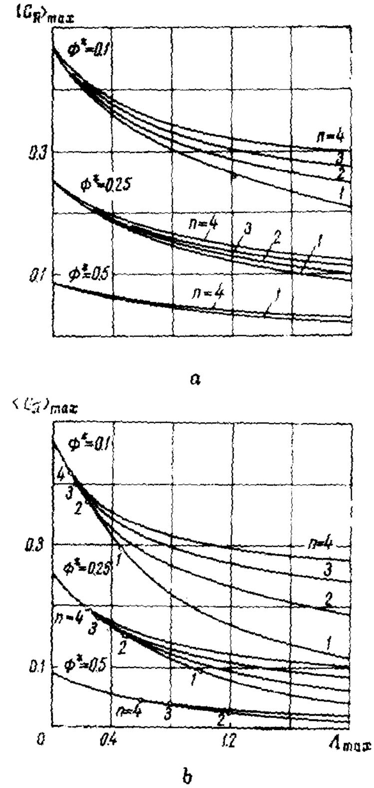

FIG. 10a, b

Λ_{max}) and on Fig. 10a and b (maximum payload $\langle G \rangle_\pi$ as a function of Λ_{max}). Figures 9a and 10a refer to the first case while 9b and 10b, to the second one.

5. Criterion of cost. We have said before that the problem of determining the optimal probability of a manoeuvre has a meaning only when there exists a necessity of repeating this manoeuvre many times. In this case the crtierion of cost becomes important in comparsison with the weight criterion.

In a manner analogous to (4.1) we shall introduce the mean cost of fulfilling a manoeuvre during a series consisting on n successful and m unsuccessful attempts. This is given by

$$C^{(n,m)} = n^{-1}\,[mC + (m - n)C_r] \tag{5.1}$$

where C is the cost of one attempt and C_r the loss incurred by one unsuccessful attempt.

Mathematical expectation of the cost of one successfully completed manoeuvre is equal to

$$\langle C\rangle = (R_0R)^{-1}\,[C + (1 - R_0R)C_r] \tag{5.2}$$

We assume that the cost of one attempt

$$C = C_0 + C_\varkappa + C_\mu + C_e$$

is sum of the following components: cost of placing the vehicle in the initial orbit $C_0 = c_0G_0$ is taken to be proportional to the initial weight $G_0 = G_\varkappa + G_{\mu 0} + G_\pi$). $C_\varkappa = c_\varkappa G_\varkappa$ which is the cost of the engine taken to be proportional to the weight of the engine $G_\varkappa$, cost of the propellant and its containers $C_\mu = c_\mu G_{\mu 0}$ which is taken to be proportional to the initial weight of the propellant $G_{\mu 0}$ and the losses C_e independent of weight parameters.

Loss due to one unsuccessful attempt $C_r = c_\pi G_\pi$ we consider to be equal to the cost of the payload and proportional to its weight G. Coefficients c_0, $c_\varkappa$, c_μ, c_π and C_e are assumed to be constant and known.

As a criterion of optimality we shall use the minimum of mathematical expectation of the cost of successful delivery of the payload of unit weight $c = \langle C\rangle / G_\pi$ or, if we take into account the relationship between the cost and weight,

$$c = \frac{1}{R_0RG_\pi}\,[c_0G_0 + C_e + c_\varkappa G_\varkappa + c_\mu G_{\mu 0} + (1 - R_0R)\,c_\pi G_\pi] \tag{5.3}$$

Initial weight G_0 and reliability R_0 are known, hence we can replace (5.3) by the following functional characterising the cost of completing a successful manoeuvre

$$S = \frac{1 + s_\varkappa G_\varkappa + s_\mu G_{\mu 0}}{R\,(1 - G_\varkappa - G_{\mu 0})} \tag{5.4}$$

Here the weight $G_\varkappa$ and $G_{\mu 0}$ are in terms of the initial weight G_0, while the coefficients $s_\varkappa$ and s_μ are given by

$$s_\varkappa = \frac{c_\varkappa - c_\pi}{c_0 + c_\pi + C_e / G_0} > -1, \qquad s_\mu = \frac{c_\mu - c_\pi}{c_0 + c_\pi + C_e / G_0} > -1 \tag{5.5}$$

The functional (5.4) is connected with (5.3) by:

$$S = \frac{R_0 (1 + c)}{c_0 + c_\pi + C_e / G_0}, \quad \text{or} \quad c = \frac{S}{R_0}\left(c_0 + c_\pi + \frac{C_e}{G_0}\right) - 1 \tag{5.6}$$

i.e. the minimum of S is equivalent to the minimum of c.

With reliability R and weight $G_\varkappa$ of the engine both known, it is a necessary condition for the cost S to be minimal, that the weight of the propellant $G_{\mu 0}$ is also minimal. This is obvious from the physical point of view, but for the sake of completeness we shall give the partial derivative of (5.4) with respect to $G_{\mu 0}$, which is

$$\frac{\partial S}{\partial G_{\mu 0}} = \frac{1 + s_\mu + (s_\varkappa - s_\mu) G_\varkappa}{R (1 - G_\varkappa - G_{\mu 0})^2}$$

The numerator of the above expression is always positive since

$$1 + s_\mu + (s_\varkappa - s_\mu) G_\varkappa \geqslant \begin{cases} 1 + s_\mu \text{ when } s_\varkappa - s_\mu \geqslant 0 \\ 1 + s_\varkappa \text{ when } s_\varkappa - s_\mu \leqslant 0 \end{cases} \quad (0 < G_\varkappa < 1)$$

and $1 + s_\mu > 0 \quad 1 + s_\varkappa > 0$ **by (5.5). Therefore** $\partial S / \partial G_{\mu 0} > 0$ **and minimum of S is reached when $G_{\mu 0}$ is minimum (R and $G_\varkappa$ are fixed).**

It follows then, that the results obtained in the minimal fuel problem can be utilised in determining the minimum cost S, when the reliability and weight of the engine are fixed. Optimal values of R and $G_\varkappa$ will however be different from those found from the condition of maximum payload (4.2).

The variational problem on the minimum cost S can be stated analogously to (4.3), if its first two equations with their boundary conditions and the fourth equation are replaced by

$$S^{\cdot} = e^{\Lambda}\left[\lambda \frac{1 + s_\varkappa G_\varkappa + s_\mu \Delta G_\mu}{1 - G_\varkappa - \Delta G_\mu} + q \frac{1 + s_\mu + (s_\varkappa - s_\mu) G_\varkappa}{(1 - G_\varkappa - \Delta G_\mu)^2}\right], \qquad S(0) = \frac{1 + s_\varkappa G_\varkappa}{1 - G_\varkappa} \tag{5.7}$$

$$S(T) = \min, \quad \Delta G_\mu^{\cdot} = q \quad (\Delta G_\mu(0) = 0,\ \Delta G_\mu(T) = \text{opt}), \quad \mathbf{v}^{\cdot} = P(1 - \Delta G_\mu)^{-1}\mathbf{i} + \mathbf{g}$$

Here G_μ (instantaneous reserve of the propellant) is replaced by a new phase coordinate $\Delta G_\mu = G_{\mu 0} - G_\mu(t)$ (mass exhausted up to the time t).

Let us consider once more the case of an ideally regulated propulsion system of limited power. Integration of the equation for the loss of mass due to thrust [2], gives

$$G_{\mu 0} = \frac{\Phi}{\Phi + G_\nu} \qquad \left(\Phi = \frac{\alpha}{2g} J(\varkappa)\right) \tag{5.8}$$

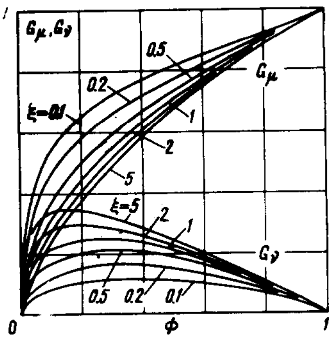

FIG. 11

where G_ν is the weight of the engine which, in this case, defines the total weight of the propulsion system $G_\varkappa$.

If the density of the sequence of failures does not depend on the weight of the engine (see (2.1) (2.12)), then the optimal value of G_ν can, as before [2], be found without solving the dynamic part of the problem on the minimum of the functional J. By putting (5.8) into (5.4), we shall be able to determine the optimal value of G_ν using the conditions $\partial S/\partial G_\nu = 0$

$$G_\nu = [\xi\Phi + \xi(\xi-1)\Phi^2]^{1/2} - \xi\Phi \qquad \left(\xi = \frac{1+s_\mu}{1+s_\nu}\right) \quad (5.9)$$

where s_ν replaces $s_\varkappa$)

Parameter ξ gives the cost ratio of the unit weight of propellant to the unit weight of the engine (see (5.5)). When $\xi = 1$ (equal costs), the formula (5.9) reduces to the previously obtained optimal relationship between the weight of propellant left and the weight of the engine. This relationship secures the maximum payload [2].

When we consider it in terms of minimum cost we find that it shifts towards the increase in the reserve of the propellant when $\xi < 1$, and towards increased weight of the engine when $\xi > 1$ (see Fig. 11). Differences between the optimal weight ratios can, when we change from the weight to the cost criterion, be very considerable and will depend on the relative cost of the propellant and the engine.

We shall now investigate the effect obtained on using weight ratios optimal for one functional, in calculating the other.

In order to reduce the number of parameters, let us eliminate from the cost functional S, the part dependent on ξ and Φ only

$$\rho(\xi, \Phi) = \frac{\xi G_{\mu 0} + G_\nu}{1 - G_{\mu 0} - G_\nu} \quad \left(S = \frac{1}{R}\left[(1 + s_\nu)\rho + 1\right]\right) \qquad (5.10)$$

The magnitude ρ characterises, similarly to S, the cost of completing a manoeuvre without however taking reliability into account.

Figure 12 shows, in solid lines, the relationship $\rho(\xi, \Phi)$ when values, optimal with respect to cost, of (5.9) and (5.8) are used for $G_{\mu 0}$ and G_ν. Dotted lines refer to the case when the values

$$G_{\mu 0} = \sqrt{\Phi}, \qquad G_\nu = \sqrt{\Phi} - \Phi$$

optimal for the weight criterium are taken for $G_{\mu 0}$ and G_ν (see (5.8) and (5.9) for $\xi = 1$). Fig. 12 shows also the dependence of the payload

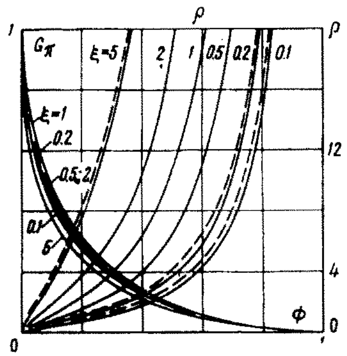

FIG. 12

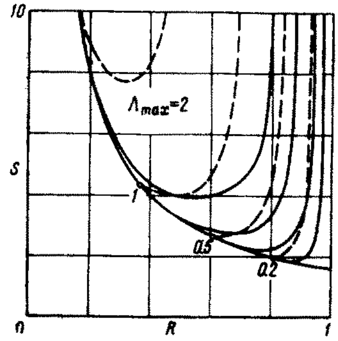

FIG. 13

$$G_\pi = 1 - G_\nu - G_{\mu 0}$$

on Φ, for various values of ξ. It is clear that the loss in the payload on changing from one set of optimal weight ratios to the other is higher, than the loss in the cost (solid and dotted lines depicting the cost ρ coincide for $\xi = 0.5, 1, 2$).

In conclusion we shall show how the optimal probability of completing a manoeuvre is affected by adopting the cost criterion instead of the weight criterion. We shall again use the relation J ($\varkappa$) from Sect. 4. Fig. 13 gives the graph S (R) (cost versus reliability) for the same parametric values as on Fig. 8 (again, solid lines denote translation between two points at rest, dotted lines – velocity increment). It is easily seen that the optimal values of the probability of completing a manoeuvre are somewhat higher than those obtained when the weight criterium was used (compare the positions of minima on Fig. 13 with the positions of maxima on Fig. 8). This can be caused by the fact that in minimising the cost, we take into account an additional loss incurred during an unsuccessful attempt of completing a manoeuvre, the loss being equal to the cost of the payload.

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